Arch. Math. 107 (2016), 227–237

0003-889X/16/030227-11
published online July 6, 2016
DOI 10.1007/s00013-016-0938-7

Archiv der Mathematik

A note on Seshadri constants of line bundles on hyperelliptic surfaces

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Abstract. We study the Seshadri constants of ample line bundles on hyperelliptic surfaces. We obtain new lower bounds and compute the exact values of the Seshadri constants in some cases. Our approach uses results of Serrano (Math. Z. 203:527–533, 1990), Harbourne and Roé (J. Pure Appl. Alg. 212:616–627, 2008), Bastianelli (Manuscripta Math. 130:113–120, 2009), Knutsen, Syzdek and Szemberg (Math. Res. Lett. 16:711–719, 2009).

Mathematics Subject Classification. 14C20.

Keywords. Seshadri constants, Hyperelliptic surfaces, Xu-type lemma.

1. Introduction. Seshadri constants measure how positive a line bundle is. They were introduced in 1992 by Demailly [9] as an attempt to tackle the famous Fujita conjecture. The conjecture has not been proven but Seshadri constants soon became an object of study on their own.

Giving exact values or just estimating Seshadri constants is very hard, even in case of line bundles on algebraic surfaces, see, e.g., [4]. There exists an upper bound for the Seshadri constant of a line bundle at points $x_1, \ldots, x_r$ on a smooth projective n-dimensional variety X, namely $\varepsilon(L, x_1, \ldots, x_r) \leq \sqrt[n]{\frac{L^n}{r}}$. Regarding lower bounds, there are examples due to Miranda and Viehweg which show that the Seshadri constants of an ample line bundle can attain arbitrarily small positive values.

Let us now recall some results concerning Seshadri constants on surfaces with Kodaira dimension zero. In the appendix to [3], Bauer and Szemberg give an upper bound for the global Seshadri constant of an ample line bundle on an abelian surface and as a corollary they obtain that the Seshadri constant of such a line bundle is always rational. In [2] Bauer computes the Seshadri constants on all $K3$ surfaces of degree 4. This result is extended by Galati and

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Knutsen [11] who computed the Seshadri constants on $K3$ surfaces of degrees 6 and 8. Earlier in [14] Knutsen estimates the Seshadri constants on $K3$ surfaces with Picard number 1. Szemberg [18] proves that the global Seshadri constants on Enriques surfaces are always rational and also provides the lower bound for Seshadri constant at an arbitrary point.

Our main result (Theorems 3.4 and 3.5), giving an estimate on the global Seshadri constant of an ample line bundle on a hyperelliptic surface, is the following

Main Theorem. *Let S be a hyperelliptic surface. Let L be an ample line bundle of type (a, b) on S. Then*

$$\varepsilon(L) \geq \min\{a, b\}.$$

Moreover, if S is a hyperelliptic surface of type 1, then $\varepsilon(L) = \min\{a, b\}$.

The paper is organised in the following way: in Theorem 3.1 we compute the global Seshadri constant of a line bundle of type (1,1) on a hyperelliptic surface of an arbitrary type. In Proposition 3.3 we point out a hyperelliptic surface type and a point at which the Seshadri constant of a line bundle of type (1,1) is strictly greater than 1. In Theorem 3.4 we compute the global Seshadri constant of an arbitrary ample line bundle on a hyperelliptic surface of type 1, and in Theorem 3.5 we provide a lower bound for the global Seshadri constant on hyperelliptic surfaces of types 2–7. Finally, in Theorem 3.6 we estimate from below the multi-point Seshadri constant of an ample line bundle at r very general points on hyperelliptic surfaces.

2. Notation and auxiliary results. Let us set up the notation and basic definitions. Our surfaces are always smooth irreducible projective varieties of dimension 2 defined over the field of complex numbers $\mathbb{C}$, curves are irreducible subvarieties of dimension 1. By $D_1 \equiv D_2$ we denote the numerical equivalence of divisors D_1 and D_2. We use the notation as in [16].

2.1. Seshadri constants. Let X be a smooth projective variety and L a nef line bundle on X. We recall the definition of the Seshadri constant.

Definition 2.1. (1) The Seshadri constant of L at a given point $x \in X$ is the real number

$$\varepsilon(L, x) = \inf\left\{\frac{LC}{\operatorname{mult}_x C} : C \ni x\right\},$$

where the infimum is taken over all curves $C \subset X$ passing through x.

(2) The global Seshadri constant of L is defined to be

$$\varepsilon(L) = \inf_{x \in X} \varepsilon(L, x).$$

Let $x_1, \ldots, x_r$ be pairwise distinct points. The notion of the Seshadri constant of a line bundle at a point may be generalised to r points in the following way:

Definition 2.2. The multi-point Seshadri constant of L at $x_1, \ldots, x_r$ is the real number

$$\varepsilon(L, x_1, \ldots, x_r) = \inf \left\{ \frac{LC}{\sum_{i=1}^r \operatorname{mult}_{x_i} C} : \ \{x_1, \ldots, x_r\} \cap C \neq \emptyset \right\},$$

where the infimum is taken over all curves $C \subset X$ passing through at least one of the points $x_1, \ldots, x_r$.

For a fixed line bundle L, the function $(x_1, \ldots, x_r) \mapsto \varepsilon(L, x_1, \ldots, x_r)$ is constant for points in very general position; moreover, its value for points in very general position is equal to $\sup\{\varepsilon(L, x_1, \ldots, x_r)\}$ where the supremum is taken over all choices of r distinct points $x_1, \ldots, x_r \in X$ (see [16, Example 5.1.11]). We denote the Seshadri constant of L at r points in very general position by $\varepsilon(L, r)$.

Let $\alpha_0(L, m_1, \ldots, m_r)$ be the least degree LC of a curve C passing through r points in general position with multiplicities $m_1, \ldots, m_r$. Let $m^{[l]} = \underbrace{(m, \ldots, m)}_{l \text{ times}}$. Then the following theorem holds:

Theorem 2.3 (Harbourne, Roé, [13, Theorem 1.2.1]). *Let L be a big and nef line bundle on a smooth projective surface. Let $r \in \mathbb{N}$, $r \geq 2$, let $\mu \in \mathbb{R}$, $\mu \geq 1$. If*

(1) *for every $m \in \mathbb{N}$ such that $1 \leq m < \mu$, we have $\alpha_0(L, m^{[r]}) \geq m\sqrt{L^2\left(r - \frac{1}{\mu}\right)}$, and*
(2) *for every $m \in \mathbb{N}$ such that $1 \leq m < \frac{\mu}{r-1}$ and if for every $k \in \mathbb{Z}$ such that $k^2 < \frac{r}{r-1}\min\{m, m+k\}$ we have $\alpha_0(L, m^{[r-1]}, m+k) \geq \frac{mr+k}{r}\sqrt{L^2\left(r - \frac{1}{\mu}\right)}$, then*

$$\varepsilon(L, r) \geq \sqrt{\frac{L^2}{r}}\sqrt{1 - \frac{1}{r\mu}}.$$

For background on Seshadri constants, we refer to an interesting overview [5].

2.2. Hyperelliptic surfaces. Let us start with recalling the definition of a hyperelliptic surface.

Definition 2.4. A hyperelliptic surface S (sometimes called bielliptic) is a surface with Kodaira dimension equal to 0 and irregularity $q(S) = 1$.

Alternatively ([7, Definition VI.19]), a surface S is hyperelliptic if $S \cong (A \times B)/G$, where A and B are elliptic curves, and G is an abelian group acting on A by translation and acting on B, such that A/G is an elliptic curve and $B/G \cong \mathbb{P}^1$; G acts on $A \times B$ coordinatewise. Hence we have the following situation:

$$\begin{array}{ccc} S \cong (A \times B)/G & \xrightarrow{\Phi} & A/G \\ \Psi \downarrow & & \\ B/G \cong \mathbb{P}^1 & & \end{array}$$

where Φ and Ψ are the natural projections.

Hyperelliptic surfaces were classified at the beginning of the 20th century by Bagnera and de Franchis [8], and independently by Enriques and Severi [10]. They showed that there are seven non-isomorphic types of hyperelliptic surfaces. These types are characterised by the action of G on $B \cong \mathbb{C}/(\mathbb{Z}\omega \oplus \mathbb{Z})$ (for details see, e.g., [7, VI.20]). The canonical divisor K_S of each hyperelliptic surface is numerically trivial.

In 1990 Serrano [17] characterised the group $\mathrm{Num}(S)$ for each type of surface:

Theorem 2.5 (Serrano). *A basis of the group of classes of numerically equivalent divisors* $\mathrm{Num}(S)$ *for each of type of surface and the multiplicities of the singular fibres in each case are the following:*

Type of a hyperelliptic surface	G	$m_1, \ldots, m_s$	*Basis of* $\mathrm{Num}(S)$
1	$\mathbb{Z}_2$	$2, 2, 2, 2$	$A/2, B$
2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$2, 2, 2, 2$	$A/2, B/2$
3	$\mathbb{Z}_4$	$2, 4, 4$	$A/4, B$
4	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$2, 4, 4$	$A/4, B/2$
5	$\mathbb{Z}_3$	$3, 3, 3$	$A/3, B$
6	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$3, 3, 3$	$A/3, B/3$
7	$\mathbb{Z}_6$	$2, 3, 6$	$A/6, B$

Let $\mu = \mathrm{lcm}\{m_1, \ldots, m_s\}$ and let $\gamma = |G|$. Notice that a basis of $\mathrm{Num}(S)$ consists of divisors A/μ and $(\mu/\gamma)\, B$.

Definition 2.6. We say that L is a line bundle of type (a, b) on a hyperelliptic surface, or $L \equiv (a, b)$ for short, if $L \equiv a \cdot A/\mu + b \cdot (\mu/\gamma)B$.

In $\mathrm{Num}(S)$ we have $A^2 = 0$, $B^2 = 0$, $AB = \gamma$. Due to [1, Proposition 5.2], we have a criterion for effectiveness of a divisor of type $(0, b)$, i.e. a divisor numerically equivalent to $b \cdot (\mu/\gamma)\, B$, namely

Lemma 2.7. *Let D be a divisor of type $(0, b)$, $b \in \mathbb{Z}$, on a hyperelliptic surface S. Then*

$$D \textit{ is effective if and only if } b \cdot (\mu/\gamma) \in \mathbb{N}.$$

The following proposition holds:

Proposition 2.8 (see [17, Lemma 1.3]). *Let D be a divisor of type (a, b) on a hyperelliptic surface S. Then*

1. *$\chi(D) = ab$;*
2. *D is ample if and only if $a > 0$ and $b > 0$;*
3. *If D is ample, then $h^0(D) = \chi(D) = ab$.*

2.3. Bounds on the self-intersection of curves. The adjunction formula, applied to the normalisation of a curve, implies the following bound for the self-intersection of a curve:

Remark 2.9 (Genus formula, [12, Lemma, p. 505]). *Let C be a curve on a surface S, passing through x_1, ..., x_r with multiplicities, respectively, m_1, ..., m_r. Let $g(C)$ denote the genus of the normalisation of C. Then*

$$g(C) \leq \frac{C^2 + C.K_S}{2} + 1 - \sum_{i=1}^{r} \frac{m_i(m_i-1)}{2}.$$

Note that:

Remark 2.10. Each curve C on a hyperelliptic surface has genus at least 1. Otherwise the normalisation of C, of genus zero, would be a covering (via Φ) of the elliptic curve A/G. This contradicts the Riemann–Hurwitz formula.

For families of curves, we have a Xu-type lemma. The original version of this lemma was proved by Xu [19]. We will use the generalisation of Xu's Lemma obtained by Knutsen, Syzdek, Szemberg [15], and independently by Bastianelli [6]. Let $\mathrm{gon}(C)$ denote the gonality of a smooth curve C, i.e. the minimal degree of a covering $C \to \mathbb{P}^1$.

Lemma 2.11 (Bastianelli, [6, Lemma 2.2]; Knutsen–Syzdek–Szemberg, [15, Theorem A]). *Let S be a smooth projective surface. Let U be a smooth variety. Consider a nontrivial family $\{(C_u, x_u)\}_{u\in U}$ where x_u is a very general point of S and C_u is a curve satisfying the condition $\mathrm{mult}_{x_u} C_u \geq m$ for every $u \in U$ and for some integer $m \geq 2$. Then for a general curve C of this family*

$$C^2 \geq m(m-1) + \mathrm{gon}(\widetilde{C}).$$

Applying the Xu-type lemma to a family $\mathcal{C}$ of curves passing through x_1, ..., x_r with multiplicities, respectively, m_1, ..., m_r, where $m_1 \geq 2$, on a blow-up at x_2, ..., x_r, we have the following multi-point version of the Xu-type lemma.

Lemma 2.12. *For a general curve C of the family $\mathcal{C}$ as above, we have*

$$C^2 \geq \left(\sum_{i=1}^{r} m_i^2\right) - m_1 + \mathrm{gon}(\widetilde{C}).$$

By Remark 2.10 there are no rational curves on a hyperelliptic surface S hence for every curve $C \subset S$ on we have $\mathrm{gon}(\widetilde{C}) \geq 2$.

3. Main results.

3.1. Seshadri constants of ample line bundles on hyperelliptic surfaces. We start with computing the global Seshadri constant in the simplest case of an ample line bundle on a hyperelliptic surface, i.e. for a line bundle of type $(1, 1)$.

Theorem 3.1. *Let S be a hyperelliptic surface. Let L be a line bundle of type $(1, 1)$ on S. Then*

$$\varepsilon(L) = 1.$$

Proof. Let $C \equiv (\alpha, \beta)$ denote a curve passing through a point $x \in S$ with multiplicity m, $m \geq 1$. We estimate the value of $\frac{LC}{m}$ from below.

Depending on the position of the point x and on the type of the hyperelliptic surface, we have the following possibilities for C to be a curve:

(1) $C \equiv B \equiv (0, k)$ and x is an arbitrary point, where $k = 1$ for a hyperelliptic surface of an odd type; $k = 2$ for a hyperelliptic surface of type 2 and 4; $k = 3$ for a hyperelliptic surface of type 6 (for admissible values of k see Theorem 2.5 and Lemma 2.7). Then

$$\frac{LC}{m} = \frac{k}{1} \geq 1.$$

(2) $C \equiv nA/\mu \equiv (n, 0)$ and the point x lies on a fibre nA/μ, where $n \in \{1, 2\}$ for a hyperelliptic surface of type 1 and 2; $n \in \{1, 2, 4\}$ for type 3 and 4; $n \in \{1, 3\}$ for type 5 and 6; $n \in \{1, 2, 3, 6\}$ for type 7 (for admissible values of n see Theorem 2.5). Then

$$\frac{LC}{m} = \frac{n}{1} \geq 1.$$

(3) $C \equiv (\alpha, \beta)$, where $\alpha > 0$ and $\beta > 0$, and x is an arbitrary point. Then by Bézout's theorem, intersecting C with a fibre B and with an appropriate fibre nA/μ depending on the position of the point x, we get:

$$\frac{LC}{m} = \frac{\alpha + \beta}{m} \geq \begin{cases} 1 & \text{in case of a hyperelliptic surface of type } 1, 3, 5, 7; \\ \frac{1}{2} + \frac{1}{2} & \text{in case of a hyperelliptic surface of type } 2; \\ \frac{1}{2} + \frac{1}{4} & \text{in case of a hyperelliptic surface of type } 4; \\ \frac{1}{3} + \frac{1}{3} & \text{in case of a hyperelliptic surface of type } 6. \end{cases}$$

Therefore $\frac{LC}{m} \geq 1$ for a hyperelliptic surface of type 1, 2, 3, 5, and 7.

Now let S be a surface of type 4 or 6. We consider two cases. If $m = 1$, then $\frac{LC}{m} = \frac{\alpha+\beta}{m} \geq \frac{2}{1} > 1$. If $m \geq 2$, then by the genus formula $C^2 \geq m^2 - m$, by the Hodge index theorem $(LC)^2 \geq L^2C^2 = 2C^2 \geq 2(m^2 - m)$, and therefore $\frac{LC}{m} \geq \sqrt{\frac{2(m^2-m)}{m^2}} = \sqrt{2 - \frac{2}{m}} \geq 1$.

Hence, independently of the type of the hyperelliptic surface, we have $\varepsilon(L, x) \geq 1$. Moreover, for each hyperelliptic type of surface, $\varepsilon(L, x) = 1$ for a point x on a fibre A/μ. Therefore $\varepsilon(L) = 1$. □

From the proof of Theorem 3.1, we immediately obtain the following corollary:

Corollary 3.2. *Let S be a hyperelliptic surface of an odd type. Let L be a line bundle of type $(1, 1)$ on S. Then the Seshadri constant of L at any $x \in S$ is computed by a fibre B, hence*

$$\varepsilon(L, x) = 1 \textit{ for any } x \in S.$$

On the other hand, it is not true that on each hyperelliptic surface the equality $\varepsilon(L, x) = 1$ holds for every $x \in S$.

Proposition 3.3. *There exists a hyperelliptic surface S such that for a line bundle L of type* $(1,1)$

$$\varepsilon(L,1) > 1.$$

Proof. Let S be a hyperelliptic surface of type 2, and let L be a line bundle of type $(1,1)$ on S. Let x be a very general point on S. We will prove that $\varepsilon(L,x) \geq \frac{4}{3}$.

Let $C \equiv (\alpha,\beta)$ be a curve passing through x with multiplicity m, $m \geq 1$.

Let $m = 1$. Assume that $\frac{LC}{m} < \frac{4}{3}$. Then $LC < \frac{4}{3}$, hence $\alpha+\beta < \frac{4}{3}$ and thus, as α and β are nonnegative integers, $\alpha+\beta \leq 1$. Therefore either $(\alpha,\beta) \equiv (1,0) \equiv A/2$ or $(\alpha,\beta) \equiv (0,1) \equiv B/2$. Since x is a very general point, it does not lie on a singular fibre $A/2$; a divisor $B/2$ is not effective on a hyperelliptic surface of type 2 (see Lemma 2.7), a contradiction.

Now let $m \geq 2$. We have to prove that $\frac{LC}{m} \geq \frac{4}{3}$. Both sides are nonnegative, hence equivalently $(LC)^2 \geq \frac{16}{9}m^2$. By the Hodge index theorem, it is enough to show that $L^2C^2 \geq \frac{16}{9}m^2$. By the Xu-type lemma (Lemma 2.11), we have $C^2 \geq m^2 - m + 2$. Hence it is enough to prove that $2m^2 - 2m + 4 \geq \frac{16}{9}m^2$. Equivalently $(m-3)(m-6) \geq 0$. The inequality is satisfied for $m \neq 4,5$. We consider these two cases separately.

Let $m = 4$. Suppose that $\frac{LC}{4} < \frac{4}{3}$. Hence $LC < \frac{16}{3}$, so $\alpha+\beta \leq 5$. On the other hand, by the Xu-type lemma $2\alpha\beta = C^2 \geq m^2 - m + 2 = 14$, a contradiction.

For $m = 5$, if $\frac{LC}{5} < \frac{4}{3}$, then $\alpha+\beta \leq 6$. By the Xu-type lemma $\alpha\beta \geq 11$, a contradiction. This completes the proof. □

Using the same method as presented in Proposition 3.3, one can show that for a very general point x on a hyperelliptic surface of type 2 and for L of type $(1,1)$, the Seshadri constant of L at x is greater than a constant slightly bigger than $\frac{4}{3}$. The proof splits into a large number of cases, and therefore we have decided to not present it here. However, precise study of this example might support the idea that this Seshadri constant is irrational.

Now we will prove a lower bound for the global Seshadri constant of an arbitrary ample line bundle on hyperelliptic surface of type 1.

Theorem 3.4. *Let S be a hyperelliptic surface of type 1. Let L be an ample line bundle of type (a,b) on S. Then*

$$\varepsilon(L) = \min\{a,b\}.$$

Proof. Let S be a hyperelliptic surface of type 1, let $L \equiv (a,b)$. Let $C \equiv (\alpha,\beta)$ denote a curve passing through a given point x with multiplicity m, $m \geq 1$. Using Bézout's theorem, we obtain

$$\frac{LC}{m} = \frac{a\beta + b\alpha}{m} \geq \begin{cases} a & \text{if } C \equiv B \text{ and } x \text{ is an arbitrary point;} \\ b & \text{if } C \equiv A/2 \text{ and } x \text{ lies on a singular fibre } A/2; \\ 2b & \text{if } C \equiv A \text{ and } x \text{ lies on a fibre } A; \\ a+b & \text{if } C \equiv (\alpha,\beta) \text{ and } x \text{ lies on one of singular fibres } A/2; \\ \frac{a}{2}+b & \text{if } C \equiv (\alpha,\beta) \text{ and } x \text{ lies on one of the fibres } A. \end{cases}$$

Hence if x lies on a singular fibre $A/2$, then $\varepsilon(L,x) = \min\{a,b,a+b\} = \min\{a,b\}$, and if x does not lie on any singular fibre $A/2$, then $\varepsilon(L,x) = \min\{a,2b,\frac{a}{2}+b\} = \min\{a,2b\}$. Since $L\cdot(A/2) > 0$, the assertion is proved. □

By the theorem above we see that on a hyperelliptic surface of type 1 the global Seshadri constant of an ample line bundle L is always submaximal, i.e. smaller than $\sqrt{L^2}$.

Note that the method used in Theorem 3.4 does not work on hyperelliptic surfaces of other types. For hyperelliptic surfaces of type 1, the lower bound of $\frac{LC}{m}$, where a curve C is not a fibre, is always greater than the value of $\frac{LC}{m}$ for some fibre C. It is also easy to show for which fibre and for which point position the global Seshadri constant is actually reached. This is not the case for hyperelliptic surfaces of types 2–7.

For hyperelliptic surfaces of types 2–7, we have the following lower bound for the global Seshadri constant

Theorem 3.5. *Let S be a hyperelliptic surface of type greater than 1. Let L be an ample line bundle of type (a,b) on S. Then*

$$\varepsilon(L) \geq \min\{a,b\}.$$

Proof. We have that $L \equiv (a,b) \equiv \min\{a,b\}\cdot M + N$, where $M \equiv (1,1)$ and N is nef. By the definition of the Seshadri constant, for every $x \in S$

$$\varepsilon(L,x) \geq \min\{a,b\}\cdot\varepsilon(M,x) + \varepsilon(N,x) \geq \min\{a,b\}\cdot\varepsilon(M,x).$$

Hence by Theorem 3.1

$$\varepsilon(L) \geq \min\{a,b\}\cdot\varepsilon(M) = \min\{a,b\}.$$

□

3.2. Multi-point Seshadri constants of ample line bundles on non-rational surfaces. In this section we present a lower bound for Seshadri constant at r points in very general position on hyperelliptic surfaces.

The lower bound for multi-point Seshadri constants obtained in Theorem 3.6 is not far from the upper bound. As mentioned before, it is well known (see, e.g., [5, Proposition 2.1.1]) that for smooth projective surfaces $\varepsilon(L,r) \leq \sqrt{\frac{L^2}{r}}$. The Biran–Nagata–Szemberg conjecture says that for any algebraic surface there exists $r_0 > 0$ such that for every $r > r_0$, in fact, there is an equality $\varepsilon(L,r) = \sqrt{\frac{L^2}{r}}$.

Theorem 3.6. *Let S be a hyperelliptic surface. Let L be an ample line bundle on S. Then*

$$\varepsilon(L,r) \geq \sqrt{\frac{L^2}{r}}\sqrt{1-\frac{1}{8r}}, \quad r \geq 2.$$

Proof. The claim follows immediately from the Harbourne–Roé theorem (Theorem 2.3) with $\mu = 8$. The point is to check that the assumptions of the theorem are satisfied with this particular constant. Turning into details, we need to check the following two conditions:

(1) for every integer $1 \leq m < 8$, $\alpha_0(L, m^{[r]}) \geq m\sqrt{L^2\left(r-\frac{1}{8}\right)}$;

(2) for every integer $1 \leq m < \frac{8}{r-1}$ and for every integer k with $k^2 < \frac{r}{r-1}\min\{m, m+k\}$, $\alpha_0(L, m^{[r-1]}, m+k) \geq \frac{mr+k}{r}\sqrt{L^2\left(r-\frac{1}{8}\right)}$.

Ad (1). For $m = 1, 2, \ldots, 7$, we ask whether the inequality $\alpha_0(L, m^{[r]}) \geq m\sqrt{L^2\left(r-\frac{1}{8}\right)}$ is satisfied.

Let C be a curve computing $\alpha(L, m^{[r]})$. It suffices to show that $LC \geq m\sqrt{L^2\left(r-\frac{1}{8}\right)}$. As L is ample, by the Hodge index theorem it is enough to prove that $L^2C^2 \geq m^2L^2\left(r-\frac{1}{8}\right)$.

We split the proof that $C^2 \geq m^2\left(r-\frac{1}{8}\right)$ into two cases: $m = 1$ and $m > 1$.

For $m = 1$, we have $h^0(C) = \dim|C| + 1 \geq r + 1$. Moreover, by Proposition 2.8 (3), $h^0(C) = \frac{C^2}{2}$. Hence $\frac{C^2}{2} \geq r + 1$. Therefore it is enough to show that $2r + 2 \geq r - \frac{1}{8}$. This condition is satisfied for every positive r.

Now let $2 \leq m \leq 7$. By the Xu-type lemma (Lemma 2.12), $C^2 \geq rm^2 - m + 2$. Hence it is enough to show that $rm^2 - m + 2 \geq m^2\left(r-\frac{1}{8}\right)$, which is elementary.

Ad (2). In the table below we write down all values of r, m, and k satisfying the conditions $1 \leq m < \frac{8}{r-1}$ and $k^2 < \frac{r}{r-1}\min\{m, m+k\}$.

r	$m < \frac{8}{r-1}$	possible k
2	1	1
	2	$1, -1$
	3	$1, -1, 2$
	4	$1, -1, 2$
	5	$1, -1, 2, -2, 3$
	6	$1, -1, 2, -2, 3$
	7	$1, -1, 2, -2, 3$
3	1	1
	2	$1, -1$
	3	$1, -1, 2$
4	1	1
	2	$1, -1$
5	1	1
6	1	1
7	1	1
8	1	1

We have omitted the case $k = 0$ in each row, as for $k = 0$ we have the inequality already proved in (1).

Using the Hodge index theorem, analogously to (1) the condition to check is reduced to the inequality $C^2 \geq \left(\frac{mr+k}{r}\right)^2\left(r-\frac{1}{8}\right)$, where C is a curve computing $\alpha_0(L, m^{[r-1]}, m+k)$.

Again we consider two cases: $m = 1$ and $m > 1$.

Let $m = 1$. Then $k = 1$. Since the Xu-type lemma (Lemma 2.12) implies that $C^2 \geq r + 3$, we easily obtain that $C^2 \geq \left(\frac{r+1}{r}\right)^2\left(r-\frac{1}{8}\right)$.

Let $m > 1$. Then $r \in \{2, 3, 4\}$. By the Xu-type lemma $C^2 \geq (r-1)m^2 + (m+k)^2 - m + 2$, so it is enough to show that $(r-1)m^2 + (m+k)^2 - m + 2 \geq \left(\frac{mr+k}{r}\right)^2 \cdot \left(r - \frac{1}{8}\right)$ holds. After reordering the terms, we obtain the inequality $8r^2k^2 - 8r^2m + 16r^2 + m^2r^2 + 2mrk - 8rk^2 + k^2 \geq 0$. Simple computations confirm that the last inequality is satisfied for all admissible $m > 1$, r, and k. The proof is completed. $\square$

Remark 3.7. Note that the proof of Theorem 3.6 holds also for abelian surfaces with $\rho = 1$.

Acknowledgements. The author would like to thank Halszka Tutaj-Gasińska for advice and support, and Tomasz Szemberg for many helpful discussions and improving the readability of the paper.

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Received: 25 March 2016